Quadratic equations and roots

Junior Cycle · Higher Level

Understand the idea

How to start

A quadratic contains an x^2 term. A root is an x-value that makes the expression equal to zero.

1. Move all terms to one side so the other side is zero.

2. Identify a, b and c with their signs. Try guide-number factorising when suitable factors are easy to find.

3. If factorising is not straightforward, use the quadratic formula. Check the required answer format.

Key terms & definitions

Root
A value making the polynomial equal to zero.

Discriminant
The expression $b^2 - 4ac$ used to determine the number of real roots.

Repeated root
One value occurring twice as a quadratic root.

Key ideas

• Write a quadratic equation in the form $ax^2 + bx + c = 0$, with $a \neq 0$.

• Try the guide-number method first when suitable factors are easy to find. After factorising an equation equal to zero, set each bracket equal to zero.

• The quadratic formula
$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
applies whenever $a \neq 0$. A negative discriminant means there are no real solutions.

• The discriminant $b^2 - 4ac$ is positive for two distinct real roots, zero for a repeated root and negative for no real roots.

• For integer roots p and q, an equation is $(x - p)(x - q) = 0$. Expand it if required.

• Answers requested in surd form or to two decimal places often point towards the quadratic formula. It is a reliable choice if factors are not straightforward. The wording alone does not force the formula: for $x^2 = 2$, taking square roots is quicker.

Quick reminder

• Put the equation equal to zero first.

• Factorise when possible; set each factor equal to zero.

• Otherwise use the quadratic formula and consider both signs.

Worked examples

Example 01 · Guide number, then solve each bracket

Solve $x^2 - 5x - 14 = 0$.

First identify a, b and c

$x^2 \quad -5x \quad -14 \quad = 0$ This question

$ax^2 \quad +bx \quad +c \quad = 0$ Match the form

$a = 1$ — the number multiplying x^2.

$b = -5$ — the number multiplying x, including its sign.

$c = -14$ — the number on its own, including its sign.

x^2 means $1x^2$, so $a = 1$.

Guide number : $1 \times (-14) = -14$**. Target sum** : -5**.**

Chosen pair : $(-7) \times (2) = -14$; $-7 + (2) = -5$.

Why this method?

Check for a common factor first. The guide number is $a \times c$. Find two signed numbers that multiply to $a \times c$ and add to b, then split the middle term and group.

$\text{GN} = 1 \times (-14) = -14$	**Step 1** The required product is -14 and the required sum is -5.
$(-7)(2) = -14, \quad -7 + 2 = -5$	**Step 2** The signs matter : -7 and $+2$ pass both tests.
$x^2 - 7x + 2x - 14 = 0$	**Step 3** Split $-5x$ into $-7x + 2x$.
$x(x - 7) + 2(x - 7) = 0$	**Step 4** Factor the first pair and the second pair.
$(x + 2)(x - 7) = 0$	**Step 5** Take out the shared bracket $(x - 7)$.
$x + 2 = 0 \quad \text{or} \quad x - 7 = 0$	**Step 6** A product is zero only if at least one factor is zero.

$x = -2 \quad \text{or} \quad x = 7$	**Step 7** Solve each linear equation.
$(-2)^2 - 5(-2) - 14 = 0, \quad 7^2 - 5(7) - 14 = 0$	**Step 8** Check both answers in the original equation.

Example 02 · Build an equation from roots

Find a monic quadratic equation with roots 3 and -2.

Why this method?

Put the equation equal to zero. Start with factorising when possible; otherwise choose a method that can find all real roots.

$(x - 3)(x + 2) = 0$	**Step 1** Build the factors
$x^2 - x - 6 = 0$	**Step 2** Multiply every term in one bracket by every term in the other, then collect like terms.
$9 - 3 - 6 = 0$ and $4 + 2 - 6 = 0$.	**Step 3** Check both roots

Example 03 · Quadratic equations with integer roots

Solve $x^2 - 8x + 12 = 0$ by factorising. What is the larger root?

First identify a, b and c

$x^2 \quad -8x \quad +12 \quad = 0$ This question

$ax^2 \quad +bx \quad +c \quad = 0$ Match the form

$a = 1$ — the number multiplying x^2.

$b = -8$ — the number multiplying x, including its sign.

$c = 12$ — the number on its own, including its sign.

x^2 means $1x^2$, so $a = 1$.

Why this method?

Put the equation equal to zero. Start with factorising when possible; otherwise choose a method that can find all real roots.

$\text{GN} = 1 \times 12 = 12$	**Step 1** The coefficient of x is -8, so the pair must add to -8.
$(-6)(-2) = 12, \quad -6 - 2 = -8$	**Step 2** Both numbers must be negative.
$x^2 - 6x - 2x + 12 = 0$	**Step 3** Split the middle term.
$x(x - 6) - 2(x - 6) = 0$	**Step 4** Factor each pair.
$(x - 2)(x - 6) = 0$	**Step 5** Factor the common bracket.
$x = 2 \quad \text{or} \quad x = 6$	**Step 6** Set each bracket equal to zero. The larger root is 6.

Think about it: Before moving on: why must each bracket be set to zero, not the whole expression to a value?

Example 04 · Quadratic equations with surd solutions ($x^2 = n$)

Solve for x, giving exact answers : $x^2 = 2$.

Why this method?

Put the equation equal to zero. Start with factorising when possible; otherwise choose a method that can find all real roots.

$x = \pm\sqrt{2}$	**Step 1** Take the square root of both sides, remembering the negative root
$x = \pm\sqrt{2}$	**Step 2** Leave the answer as a surd, because the root is not a whole number

Think about it: Before moving on: why are there two solutions to $x^2 = n$ when n is positive?

Example 05 · Using the quadratic formula, one line at a time

Solve $x^2 - 5x + 3 = 0$, giving exact answers.

First identify a, b and c

$x^2 \quad -5x \quad +3 \quad = 0$ This question

$ax^2 \quad +bx \quad +c \quad = 0$ Match the form

$a = 1$ — the number multiplying x^2.

$b = -5$ — the number multiplying x, including its sign.

$c = 3$ — the number on its own, including its sign.

x^2 means $1x^2$, so $a = 1$.

Why this method?

The guide number is 3. No integer pair multiplies to 3 and adds to -5. Use the quadratic formula. We will give the exact surd answers first, then decimal answers.

$a = 1, b = -5, c = 3.$	**Step 1** Compare with $ax^2 + bx + c = 0$. Include the sign of each coefficient.
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	**Step 2** Write the formula before substituting, so the whole numerator and denominator are clear.
$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(3)}}{2(1)}$	**Step 3** Substitute with brackets around the negative value of b.
$(-5)^2 - 4(1)(3) = 25 - 12 = 13$	**Step 4** Calculate the discriminant inside the root first.
$x = \frac{5 \pm \sqrt{13}}{2}$	**Step 5** Simplify the signs and denominator. The ± means we must consider both choices.
$x = \frac{5 + \sqrt{13}}{2}$ or $x = \frac{5 - \sqrt{13}}{2}$.	**Step 6** Write both exact roots. Do not round the surd when exact answers are requested.
$x \approx 4.30 \quad \text{or} \quad x \approx 0.70$	**Step 7** For two decimal places, calculate from the exact roots and round only at the end.

Example 06 · Quadratic equations with rational coefficients

Solve
$\left(\frac{4}{3}\right)x^2 + \left(\frac{9}{3}\right)x + \left(\frac{1}{3}\right) = 0$
over the real numbers. Give both exact values; the radical need not be simplified.

Why this method?

Put the equation equal to zero. Start with factorising when possible; otherwise choose a method that can find all real roots.

$4x^2 + 9x + 1 = 0$	**Step 1** Multiply every term, including the right-hand side, by 3 to clear the common denominator.
$a = 4, \quad b = 9, \quad c = 1$	**Step 2** Compare the new equation with $ax^2 + bx + c = 0$. Use the coefficients of this whole-number equation.
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	**Step 3** Write the quadratic formula.
$x = \frac{-9 \pm \sqrt{(9)^2 - 4(4)(1)}}{2(4)}$	**Step 4** Replace a, b and c, keeping the whole numerator above the fraction bar.
$x = \frac{-9 \pm \sqrt{65}}{8}$	**Step 5** Calculate the discriminant inside the root and the denominator separately.
$x = \frac{-9 - \sqrt{65}}{8} \quad \text{or} \quad x = \frac{-9 + \sqrt{65}}{8}$	**Step 6** Write both choices of sign. These are exact roots; this question allows the radical to remain unsimplified.

Think about it: Why does multiplying every term by the same non-zero denominator preserve the solutions?

Example 07 · Quadratic equations with rational coefficients — another example

Solve
$\left(\frac{5}{2}\right)x^2 + \left(\frac{8}{2}\right)x + \left(\frac{2}{2}\right) = 0$
over the real numbers. Give both exact values; the radical need not be simplified.

Why this method?

Put the equation equal to zero. Start with factorising when possible; otherwise choose a method that can find all real roots.

$5x^2 + 8x + 2 = 0$	**Step 1** Multiply every term, including the right-hand side, by 2 to clear the common denominator.
$a = 5, \quad b = 8, \quad c = 2$	**Step 2** Compare the new equation with $ax^2 + bx + c = 0$. Use the coefficients of this whole-number equation.
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	**Step 3** Write the quadratic formula.
$x = \frac{-8 \pm \sqrt{(8)^2 - 4(5)(2)}}{2(5)}$	**Step 4** Replace a, b and c, keeping the whole numerator above the fraction bar.
$x = \frac{-8 \pm \sqrt{24}}{10}$	**Step 5** Calculate the discriminant inside the root and the denominator separately.
$x = \frac{-8 - \sqrt{24}}{10} \quad \text{or} \quad x = \frac{-8 + \sqrt{24}}{10}$	**Step 6** Write both choices of sign. These are exact roots; this question allows the radical to remain unsimplified.

Example 08 · Quadratic methods and real-solution checks

Solve
$\left(\frac{1}{5}\right)x^2 + \left(\frac{1}{5}\right)x + \left(\frac{21}{20}\right) = ($
over the real numbers. Give every distinct solution exactly.

Why this method?

Complete the square to see whether a real solution is possible. First remove fractions, then make the coefficient of x^2 equal to 1.

$4x^2 + 4x + 21 = 0$	**Step 1** Multiply every term by 20, the common denominator, to clear the fractions.
$x^2 + x + \frac{21}{4} = 0$	**Step 2** Divide every term by 4 so the coefficient of x^2 is 1.
$x^2 + x = -\frac{21}{4}$	**Step 3** Subtract the constant term from both sides.
$(\frac{1}{2})^2 = \frac{1}{4}$	**Step 4** Halve the coefficient of x and square it. This is the missing constant that makes a perfect square.
$x^2 + x + \frac{1}{4} = -\frac{21}{4} + \frac{1}{4}$	**Step 5** Add that same quantity to both sides. This preserves equality.
$(x + \frac{1}{2})^2 = -5$	**Step 6** The left-hand side is now a squared bracket. Simplify the right-hand side.
No real solutions.	**Step 7** A real number squared cannot be negative. Therefore there is no real value of x that can satisfy this equation.

Think about it: Why can a quadratic equation have no real solution, and how would both a graph and a completed square show this?

Example 09 · Quadratic methods and real-solution checks — another example

Solve
$\left(\frac{1}{5}\right)x^2 + (-1)x + \left(\frac{37}{20}\right) = 0$
over the real numbers. Give every distinct solution exactly.

Why this method?

Complete the square to see whether a real solution is possible. First remove fractions, then make the coefficient of x^2 equal to 1.

$4x^2 - 20x + 37 = 0$	**Step 1** Multiply every term by 20, the common denominator, to clear the fractions.
$x^2 - 5x + \frac{37}{4} = 0$	**Step 2** Divide every term by 4 so the coefficient of x^2 is 1.
$x^2 - 5x = -\frac{37}{4}$	**Step 3** Subtract the constant term from both sides.
$(-\frac{5}{2})^2 = \frac{25}{4}$	**Step 4** Halve the coefficient of x and square it. This is the missing constant that makes a perfect square.
$x^2 - 5x + \frac{25}{4} = -\frac{37}{4} + \frac{25}{4}$	**Step 5** Add that same quantity to both sides. This preserves equality.
$(x - \frac{5}{2})^2 = -3$	**Step 6** The left-hand side is now a squared bracket. Simplify the right-hand side.
No real solutions.	**Step 7** A real number squared cannot be negative. Therefore there is no real value of x that can satisfy this equation.

Example 10 · Quadratic equations with a repeated rational root

Solve over the real numbers
$: \left(-\frac{1}{3}\right)x^2 + \left(-\frac{3}{2}\right)x + \left(-\frac{27}{16}\right) = ($
. Give all distinct solutions in simplest fractional form.

Why this method?

Put the equation equal to zero. Start with factorising when possible; otherwise choose a method that can find all real roots.

$D = \left(-\frac{3}{2}\right)^2 - 4 \times \left(-\frac{1}{3}\right) \times \left(-\frac{27}{16}\right) = \frac{9}{4} - \frac{9}{4} = ($
.

Step 1
Calculate the discriminant $D = b^2 - 4ac$

$x = -\frac{b}{(2a)} = -\frac{\left(-\frac{3}{2}\right)}{\left(2 \times \left(-\frac{1}{3}\right)\right)} = -\frac{9}{4}$
.

Step 2
A zero discriminant gives one distinct real solution, a repeated root

$\left(-\frac{1}{3}\right) \times \left(x - \left(-\frac{9}{4}\right)\right)^2 = ($
. Since $-\frac{1}{3}$ is non-zero
$, x - \left(-\frac{9}{4}\right) = 0$
. Thus $x = -\frac{9}{4}$.

Step 3
Check by rewriting as a constant times a square

Think about it: Why does the ± sign give only one distinct solution when the discriminant is zero?

Example 11 · Quadratic equations with a repeated rational root — another example

Solve over the real numbers
$: \left(\frac{2}{5}\right)x^2 + \left(-\frac{22}{5}\right)x + \left(\frac{121}{10}\right) = ($
. Give all distinct solutions in simplest fractional form.

Why this method?

Put the equation equal to zero. Start with factorising when possible; otherwise choose a method that can find all real roots.

$D = \left(-\frac{22}{5}\right)^2 - 4 \times \left(\frac{2}{5}\right) \times \left(\frac{121}{10}\right) = \frac{484}{25} - \frac{484}{25} = ($
.

Step 1
Calculate the discriminant $D = b^2 - 4ac$

$x = -\frac{b}{(2a)} = -\frac{\left(-\frac{22}{5}\right)}{\left(2 \times \left(\frac{2}{5}\right)\right)} = \frac{11}{2}$.	**Step 2** A zero discriminant gives one distinct real solution, a repeated root
$\left(\frac{2}{5}\right) \times \left(x - \left(\frac{11}{2}\right)\right)^2 = 0$. Since $\frac{2}{5}$ is non-zero , $x - \left(\frac{11}{2}\right) = 0$. Thus $x = \frac{11}{2}$.	**Step 3** Check by rewriting as a constant times a square

Example 12 · The constant term from integer roots

A quadratic equation $x^2 + bx + c = 0$ has roots 3 and 8. What is the value of c?

Why this method?

Put the equation equal to zero. Start with factorising when possible; otherwise choose a method that can find all real roots.

$(x - 3)(x - 8) = 0$	**Step 1** The roots give the factors (x) root)
$c = 3 \times 8 = 24$	**Step 2** Expand: the constant term is the product of the roots

Think about it: Before moving on: which part of the expanded quadratic comes from the sum of the roots, and which from the product?

Common mistake

Dividing only the square-root part by $2a$ in the quadratic formula. The whole numerator is divided: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Keep both terms above the same fraction bar.

Exam technique

The guide-number method can be quicker when suitable factors are easy to find. The quadratic formula is always available for a quadratic equation. Keep surds exact, or round only the final answers to the requested decimal places.

Try it yourself

Work on paper before checking the answers at the end. These questions are for self-checking, not submitted homework.

Question 1

Solve
$\left(\frac{2}{4}\right)x^2 + \left(-\frac{8}{4}\right)x + \left(-\frac{6}{4}\right) = 0$
over the real numbers. Give both exact values; the radical need not be simplified.

A.
$x = \frac{(8 - \sqrt{16})}{4}$
or
$x = \frac{(8 + \sqrt{16})}{4}$

B.
$x = \frac{(8 - \sqrt{112})}{2}$
or
$x = \frac{(8 + \sqrt{112})}{2}$

C.
$x = \frac{(8 - \sqrt{112})}{4}$
or
$x = \frac{(8 + \sqrt{112})}{4}$

D.
$x = \frac{(-8 - \sqrt{112})}{4}$
or
$x = \frac{(-8 + \sqrt{112})}{4}$

Question 2

Solve
$\left(\frac{1}{6}\right)x^2 + \left(-\frac{11}{6}\right)x + \left(\frac{133}{24}\right) = 0$
over the real numbers. Give every distinct solution exactly.

A. $x = \frac{11}{2} - \sqrt{3}$ or $x = \frac{11}{2} + \sqrt{3}$

B. $x = \frac{11}{2}$

C. All real x.

D. No real solutions.

Question 3

Solve over the real numbers $: \left(-\frac{1}{4}\right)x^2 + \left(-\frac{7}{4}\right)x + \left(-\frac{49}{16}\right) = 0$. Give all distinct solutions in simplest fractional form.

A. $x = -7$

B. $x = -\frac{7}{2}$ or $x = \frac{7}{2}$

C. $x = -\frac{7}{2}$

D. $x = \frac{7}{2}$

Question 4

Solve for x, giving exact answers : $x^2 = 31$.

A. $x = \pm\frac{31}{2}$

B. $x = \pm 31$

C. $x = \pm\sqrt{31}$

D. $x = \sqrt{31}$

Question 5

Solve $x^2 - 11x + 24 = 0$ by factorising. What is the larger root?

Question 6

A quadratic equation $x^2 +$ bx $+c = 0$ has roots 6 and 7. What is the value of c?

Watch an explanation

Corbettmaths — Quadratics: solving (factorising)

Cognito — The Quadratic Formula

Open the website chapter to load the embedded players. Use the rates and rules in your question for financial examples.

Self-check answers

Question 1

$$x = \frac{(8 - \sqrt{112})}{4}$$

or

$$x = \frac{(8 + \sqrt{112})}{4}$$

Question 2

No real solutions.

Question 3

$x = -\frac{7}{2}$

Question 4

$x = \pm\sqrt{31}$

Question 5

8

Question 6

42

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